MAGNETIC SPACECRAFT ATTITUDE STABILIZATION VIA OUTPUT FEEDBACK WITH SEPARATION BETWEEN MEASUREMENT AND ACTUATION

Fabio Celani
Sapienza University of Rome
School of Aerospace Engineering
Via Salaria, 851
00138, Roma, Italy
e-mail: fabio.celani@uniroma1.it, web page: https://sites.google.com/a/uniroma1.it/fabiocelani_eng/

Key words: Attitude Control of Spacecrafts, Magnetic Actuators, Stability.

Summary. In actual implementations of magnetic control laws for spacecraft attitude stabilization, the time in which Earth magnetic field is measured must be separated from the time in which magnetic dipole moment is generated. We propose a magnetic attitude stabilization law that takes into account of the latter separation. Specifically, a feedback that besides measurement of the geomagnetic field, needs measures of attitude only (output feedback) is presented, and a design method for determining its parameters is obtained. A case study is included to validate the effectiveness of the proposed control law.

1. INTRODUCTION

Spacecraft attitude control can be obtained by adopting several actuation mechanisms. Among them electromagnetic actuators are widely used for generation of attitude control torques on satellites flying in low Earth orbits. They consist of planar current-driven coils rigidly placed on the spacecraft typically along three orthogonal axes, and they operate on the basis of the interaction between the magnetic dipole moment generated by those coils and the Earth’s magnetic field; in fact, the latter interaction generates a torque that attempts to align the magnetic dipole moment in the direction of the field. Magnetic actuators, also known as magnetorquers, have the important limitation that control torque is constrained to belong to the plane orthogonal to the Earth’s magnetic field. As a result, sometimes a reaction wheel accompanies magnetorquers to provide full three-axis control; moreover, magnetorquers are often used for angular momentum dumping when reaction or momentum-bias wheels are employed for accurate attitude control (see [1, Chapter 7]). Lately, attitude stabilization using only magnetorquers has been considered as a feasible option especially for low-cost micro-satellites and for satellites with a failure in the main attitude control system. In such scenario many control laws have been designed, and a survey of various approaches adopted can be found in [2]; in particular, Lyapunov-based design...
has been adopted in \[3, 4, 5, 6, 7\]. In those works feedback control laws that, besides measuring the geomagnetic field, require measures of both attitude and attitude-rate (i.e. state feedback control laws) are proposed; moreover, in \[5\] and \[7\] feedback control algorithms which, besides measuring the geomagnetic field, need measures of attitude only (i.e. output feedback control algorithms) are presented, too.

All control laws presented in the cited works are continuous-time; thus, in principle they require a continuous-time measurement of the geomagnetic field and a continuous-time generation of coils’ currents. However, in practical implementations the time in which Earth’s magnetic field is measured (measurement time) has to be separated from the time in which coils’ currents are generated (actuation time). The latter separation is necessary because currents flowing in the spacecraft’s magnetic coils generate a local magnetic field that impairs an accurate measurement of the geomagnetic field; as a result, when the Earth’s magnetic field is being measured no currents must flow in the coils, and consequently no magnetic actuation is possible; on the other hand, when magnetic actuation is active, it is not possible to obtain accurate measurements of the geomagnetic field. Consequently, in practical applications a periodic switch between measurement time and actuation time is implemented. Magnetic control laws compatible with the latter constraint could be obtained by simply discretizing algorithms designed in continuous-time; however, following that approach could lead to a degradation of the performances of the attitude control system that could even lead to instability. In order to overcome such difficulty, it is important to design control laws that take into account of the separation constraint directly during the design phase. Paper \[8\] presents the design of a state feedback that fulfills the separation requirement described before. In the proposed work, the focus is on designing an output feedback that satisfies the previous constraint.

1.1 Notations

For \(x \in \mathbb{R}^n\), \(||x||\) denotes the Euclidean norm of \(x\). For \(a \in \mathbb{R}^3\), \(a^\times\) represents the skew symmetric matrix

\[
    a^\times = \begin{bmatrix}
    0 & -a_3 & a_2 \\
    a_3 & 0 & -a_1 \\
    -a_2 & a_1 & 0
    \end{bmatrix}
\]

so that for \(b \in \mathbb{R}^3\), multiplication \(a^\times b\) is equal to the cross product \(a \times b\).

2. MODELING AND CONTROL ALGORITHM

In order to describe the attitude dynamics of an Earth-orbiting rigid spacecraft, and in order to represent the geomagnetic field, it is useful to introduce the following reference frames.

1. Geocentric Inertial Frame \(\mathcal{F}_i\). A commonly used inertial frame for Earth orbits is the Geocentric Inertial Frame, whose origin is in the Earth’s center, its \(x_i\) axis is the vernal equinox direction, its \(z_i\) axis coincides with the Earth’s axis of rotation and points northward, and its \(y_i\) axis completes an orthogonal right-handed frame (see [1, Section 2.6.1]).
2. **Spacecraft body frame** \( \mathcal{F}_b \). The origin of this right-handed orthogonal frame attached to the spacecraft, coincides with the satellite’s center of mass; its axes are chosen so that the (inertial) pointing objective is having \( \mathcal{F}_b \) aligned with \( \mathcal{F}_i \).

Since the pointing objective consists in aligning \( \mathcal{F}_b \) to \( \mathcal{F}_i \), the focus will be on the relative kinematics and dynamics of the satellite with respect to the inertial frame. Let \( q = [q_1, q_2, q_3, q_4]^T = [q_v^T, q_i]^T \) with \( ||q|| = 1 \) be the unit quaternion representing rotation of \( \mathcal{F}_b \) with respect to \( \mathcal{F}_i \); then, the corresponding attitude matrix is given by

\[
C(q) = (q_4^2 - q_v^T q_v)I + 2q_v q_v^T - 2q_4 q_v^x
\]  
(2)

(see [9, Section 5.4]).

Let

\[
W(q) = \frac{1}{2} \begin{bmatrix} q_4 I + q_v^x \\ -q_v^T \end{bmatrix}
\]  
(3)

Then the relative attitude kinematics is given by

\[
\dot{q} = W(q)\omega
\]  
(4)

where \( \omega \in \mathbb{R}^3 \) is the angular rate of \( \mathcal{F}_b \) with respect to \( \mathcal{F}_i \) resolved in \( \mathcal{F}_b \) (see [9, Section 5.5.3]).

The attitude dynamics in body frame can be expressed by

\[
J\dot{\omega} = -\omega^x J\omega + T
\]  
(5)

where \( J \in \mathbb{R}^{3 \times 3} \) is the spacecraft inertia matrix, and \( T \) is the control torque expressed in \( \mathcal{F}_b \) (see [9, Section 6.4]).

The spacecraft is equipped with three magnetic coils aligned with the \( \mathcal{F}_b \) axes which generate the magnetic attitude control torque

\[
T = m_{coils} \times B_b = -B^{b,x} m_{coils}
\]  
(6)

where \( m_{coils} \in \mathbb{R}^3 \) is the vector of magnetic dipole moments for the three coils, and \( B_b \) is the geomagnetic field at spacecraft expressed in body frame \( \mathcal{F}_b \).

Let \( B_i \) be the geomagnetic field at spacecraft expressed in inertial frame \( \mathcal{F}_i \). Note that \( B_i \) varies with time at least because of the spacecraft’s motion along the orbit. Then

\[
B_b(q, t) = C(q)B_i(t)
\]  
(7)

which shows explicitly the dependence of \( B_b \) on both \( q \) and \( t \).

Grouping together equations (4) (5) (6) the following nonlinear time-varying system is obtained

\[
\begin{align*}
\dot{q} &= W(q)\omega \\
J\dot{\omega} &= -\omega^x J\omega - B^{b,x}(q, t) m_{coils}
\end{align*}
\]  
(8)
in which $m_{\text{coils}}$ is the control input.

In order to analyze and design control algorithms, it is important to characterize the time-dependence of $B^i(q,t)$ which is the same as characterizing the time-dependence of $B^i(t)$. Assume that the orbit is circular of radius $R$; then, adopting the so called dipole model of the geomagnetic field (see [10, Appendix H]) we obtain

$$B^i(t) = \frac{\mu_m}{R^3} [3((\hat{n}^i)^T \dot{r}^i(t)) \dot{r}^i(t) - \hat{n}^i]$$

In equation (9), $\mu_m = 7.746 \times 10^{15}$ Wb m is the total dipole strength (see ref. [11]), $r^i(t)$ is the spacecraft’s position vector resolved in $\mathcal{F}_i$, and $\dot{r}^i(t)$ is the vector of the direction cosines of $r^i(t)$; finally $\hat{n}^i$ is the vector of direction cosines of the Earth’s magnetic dipole moment expressed in $\mathcal{F}_i$. Here we set $\hat{n}^i = [0 \ 0 \ -1]^T$ which corresponds to having the dipole’s coevolution angle equal to $180^\circ$. Even if a more accurate value for that angle would be $170^\circ$ (see ref. [11]), here we approximate it to $180^\circ$ since this will substantially simplify future symbolic computations.

Equation (9) shows that in order to characterize the time dependence of $B^i(t)$, it is necessary to determine an expression for $r^i(t)$ which is the spacecraft’s position vector resolved in $\mathcal{F}_i$. Define a coordinate system $x_p, y_p$ in the orbit’s plane whose origin is at Earth’s center; then, the satellite’s position is clearly given by

$$x^p(t) = R \cos(nt + \phi_0)$$
$$y^p(t) = R \sin(nt + \phi_0)$$

where $n$ is the orbital rate, and $\phi_0$ an initial phase. Then, coordinates of the satellite in inertial frame $\mathcal{F}_i$ can be easily obtained from (10) using an appropriate rotation matrix which depends on the orbit’s inclination $incl$ and on the value $\Omega$ of the Right Ascension of the Ascending Node (RAAN) (see [1, Section 2.6.2]). Plugging into (9) the equations of the latter coordinates, an explicit expression for $B^i(t)$ can be obtained; it can be easily checked that $B^i(t)$ is periodic with period $2\pi/n$. Consequently system (8) is a periodic nonlinear system with period $2\pi/n$.

As stated before, the control objective is driving the spacecraft so that $\mathcal{F}_k$ is aligned with $\mathcal{F}_i$. From (2) it follows that $C(q) = I$ for $q = [q^T \ q] = \pm \bar{q}$ where $\bar{q} = [0 \ 0 \ 0 \ 1]^T$. Thus, the objective is designing control strategies for $m_{\text{coils}}$ so that $q_v \to 0$ and $\omega \to 0$.

In [7] both a static state feedback and a dynamic output feedback are presented; both feedbacks were obtained as modifications of those in [5, 6].

In this paper the focus is on the dynamic output feedback presented in [7] which does not require the measurement of $\omega$ and is given by

$$\dot{\delta}(t) = \alpha(q(t) - \epsilon \delta(t))$$
$$m_{\text{coils}}(t) = (B^k(q(t), t)^x \epsilon^2 (k_1 q_v(t) + k_2 \alpha \lambda W(q(t))^T (q(t) - \epsilon \lambda \delta(t))))$$

with $\delta \in \mathbb{R}^4$. It is shown in [7] that selecting $k_1 > 0$, $k_2 > 0$, $\alpha > 0$, $\lambda > 0$, and choosing $\epsilon > 0$ small enough, local exponentially stability of equilibrium $(q, \omega, \delta) = (\bar{q}, 0, \frac{1}{\epsilon \alpha} \bar{q})$ is achieved.
Feedback control law (11) seems implementable in practice since $B^b$ and $q$ can be measured using magnetometers and attitude sensors respectively. However, as explained in the introduction, the time in which $B^b$ is measured should be separated from the time in which magnetic control action is applied; in fact, magnetic torque is obtained by generating currents flowing through magnetic coils, and those currents create a local magnetic field which impairs an accurate measurement of $B^b$. As a result, in order to take into account of the previous constraint, we should consider only magnetic feedback laws of the following type; in a first interval of length $\Delta_m$, $B^b$ is measured (along with $q$) and $m_{coils}$ is set to 0; then, in a successive interval of length $\Delta_a$, $m_{coils}$ is generated and held constant because the measure of $B^b$ cannot be updated during that interval; the operations of measurement and actuation are repeated periodically. Thus, the considered scenario is represented by the time diagram in Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{time_diagram.png}
\caption{Time diagram (j=0,1,2,…).}
\end{figure}

Note that the value of $\Delta_m$ is determined mainly by the following physical fact; in order to measure $B^b$, no current must flow in the magnetic coils; then, when currents in the magnetorquers are switched off, it is necessary to wait for the dissipation of existing currents through the coils before obtaining an accurate measurement of $B^b$; usually the latter dissipation time is much larger than the time necessary to read and process data from magnetometers and attitude sensors; as a consequence, $\Delta_m$ is mostly determined by the dissipation phenomenon just described. Reference [12] considers the case in which $\Delta_m = 0$ which corresponds to having $\Delta_m$ negligible with respect to $\Delta_a$; here, we study situations in which that does not occur and consequently $\Delta_m > 0$.

In order to simplify forthcoming expressions let us define $\Delta \equiv \Delta_m + \Delta_a$. A dynamic output feedback law compatible with the requirement of separation between measurement and actuation can be obtained by enforcing $m_{coils}$ to zero during the measurement time and discretizing (11) using the forward differencing approximation (see ref. [13]) during the actuation time; thus, the following control law is obtained

\[
\begin{align*}
\delta((j+1)\Delta + \Delta_m) &= \delta(j\Delta + \Delta_m) + \Delta \alpha (q(j\Delta + \Delta_m) - \epsilon \lambda \delta(j\Delta + \Delta_m)) \\
m_{coils}(t) &= \begin{cases} \\
0 & j\Delta \leq t < j\Delta + \Delta_m \\
(B^b(q(j\Delta + \Delta_m), j\Delta + \Delta_m)^T) \\
\epsilon^2 (k_1 q_v(j\Delta + \Delta_m) + k_2 \alpha \lambda W(q(j\Delta + \Delta_m))} & j\Delta + \Delta_m \leq t < (j + 1)\Delta \\
(q(j\Delta + \Delta_m) - \epsilon \lambda \delta(j\Delta + \Delta_m)) & j = 0, 1, 2, \ldots
\end{cases}
\end{align*}
\]

(12)
3. Feedback Design

In this section we will focus on output-feedback law (12) and will derive conditions on parameters \( k_1, k_2, \alpha, \lambda, \epsilon \), and intervals \( \Delta_m \) and \( \Delta_n \) which ensure that equilibrium \((q, \omega, \delta) = (\bar{q}, 0, 1/\lambda \bar{q})\) is locally exponentially stable for closed-loop system (8) (12). In order to derive those conditions, it suffices considering the restriction of closed-loop system (8) (12) to the open set \( S^3 \times \mathbb{R}^7 \) where

\[
S^3 = \{ q \in \mathbb{R}^4 \mid ||q|| = 1, \; q_4 > 0 \}
\]

Since on the latter set \( q_4 = (1 - q_v^T q_v)^{1/2} \) then the considered restriction is given by the interconnection of the following systems

\[
\dot{q}_v(t) = W_v(q_v(t))\omega(t) \quad J\dot{\omega}(t) = -\omega(t)^T J\omega(t) - [C_v(q_v(t))B^i(t)]^\times m_{\text{coils}}(t)
\]

\[
\begin{align*}
\delta_v(j + 1)\Delta + \Delta_m &= \delta_v(j\Delta + \Delta_m) + \Delta \alpha (q_v(j\Delta + \Delta_m) - \epsilon \lambda \delta_v(j\Delta + \Delta_m)) \\
\hat{\delta}_4(j + 1)\Delta + \Delta_m &= \hat{\delta}_4(j\Delta + \Delta_m) + \Delta \alpha ((1 - q_v(j\Delta + \Delta_m)^T q_v(j\Delta + \Delta_m))^{1/2} - \epsilon \lambda \hat{\delta}_4(j\Delta + \Delta_m) - 1)
\end{align*}
\]

\[
m_{\text{coils}}(t) = \begin{cases} 0 & j\Delta \leq t < j\Delta + \Delta_m \\ (C_v(q_v(j\Delta + \Delta_m))B^i(j\Delta + \Delta_m)^\times)^T & \epsilon^2 (k_1 q_v(j\Delta + \Delta_m) + k_2\lambda W_r(q_v(j\Delta + \Delta_m)))^T \\ \epsilon \lambda q_v(j\Delta + \Delta_m) & j\Delta + \Delta_m \leq t < (j + 1)\Delta \\ \epsilon \lambda \hat{\delta}_4(j\Delta + \Delta_m) + 1 & j = 0, 1, 2, \ldots \end{cases}
\]

where \( \delta_v = [\delta_1 \; \delta_2 \; \delta_3]^T \), \( \hat{\delta}_4 = \delta_4 - \frac{1}{\lambda} \)

\[
W_v(q_v) = \frac{1}{2} \left[ (1 - q_v^T q_v)^{1/2} I + q_v^\times \right] \quad W_r(q_v) = \left[ \begin{array}{c} W_v(q_v) \\ -\frac{1}{2} q_v^T \end{array} \right] \quad C_v(q_v) = (1 - 2q_v^T q_v) I + 2q_v q_v^T - 2(1 - q_v^T q_v)^{1/2} q_v^\times
\]

and where (7) has been used. It is simple to show that if \((q_v, \omega, \delta_v, \hat{\delta}_4) = (0, 0, 0, 0)\) is a locally exponentially stable equilibrium for (14) (15), then \((q_v, \omega, \delta) = (\bar{q}, 0, 1/\lambda \bar{q})\) is a locally exponentially stable equilibrium for (8) (12).
It will be shown next that the closed-loop system (14) (15) can be expressed as a hybrid system of the type considered in [14]. For that purpose define the following time sequence

\[
t_k = \begin{cases} 
\frac{k}{2} \Delta & k = 0, 2, 4, \ldots \\
\frac{k-1}{2} \Delta + \Delta_m & k = 1, 3, 5, \ldots 
\end{cases}
\]  

which represent the instants that appear in Fig. 1. Introduce the following discrete-time function

\[p(k) = \frac{1}{2} (1 - (-1)^k) \quad k = 0, 1, 2, \ldots
\]

Note that \(p(k)\) is equal to 0 for \(k\) even and to 1 for \(k\) odd. Then, time sequence (16) can be expressed by the following alternative form which will be useful for the sequel

\[t_k = (1 - p(k)) \frac{k}{2} \Delta + p(k) \left( \frac{k-1}{2} \Delta + \Delta_m \right) \quad k = 0, 1, 2, \ldots
\]

By using the previous expressions, it follows that equation (15) can be written in the following compact form

\[
\begin{align*}
\delta_v(t_{k+1}) &= \delta_v(t_k) + p(k) \Delta \alpha (q_v(t_k) - \epsilon \lambda \delta_v(t_k)) \\
\tilde{\delta}_4(t_{k+1}) &= \tilde{\delta}_4(t_k) + p(k) \Delta \alpha \left( (1 - q_v(t_k)Tq_v(t_k))^{1/2} - \epsilon \lambda \tilde{\delta}_4(t_k) - 1 \right) \\
m_{\text{coils}}(t) &= p(k)(C_v(q_v(t_k))B^f(t_k)^T\epsilon^2 (k_1q_v(t_k) + k_2\alpha \lambda W_v(q_v(t_k))^T \\
&\quad \left( \left[ 1 - q_v(t_k)Tq_v(t_k) \right]^{1/2} - \epsilon \lambda \tilde{\delta}_4(t_k) + 1 \right) \\
\end{align*}
\]  

Note then that the closed-loop system (14) (19) is a nonlinear time-varying hybrid system of the type considered in [14].

Next, consider the linear approximation of (14) (19) about the equilibrium \((q_v, \omega, \delta_v, \tilde{\delta}_4) = (0, 0, 0, 0)\) as defined in [14]. The latter approximation is given by

\[
\begin{align*}
\dot{q}_v(t) &= \frac{1}{2} \omega(t) \\
J\dot{\omega}(t) &= -B^f(t)^Tm_{\text{coils}}(t_k) \quad t_k \leq t < t_{k+1}
\end{align*}
\]

\[
\begin{align*}
\delta_v(t_{k+1}) &= \delta_v(t_k) + p(k) \Delta \alpha (q_v(t_k) - \epsilon \lambda \delta_v(t_k)) \\
\tilde{\delta}_4(t_{k+1}) &= \tilde{\delta}_4(t_k) - p(k) \Delta \alpha \epsilon \lambda \tilde{\delta}_4(t_k) \\
m_{\text{coils}}(t_k) &= p(k)(B^f(t_k)^T\epsilon^2 \left( k_1q_v(t_k) + \frac{1}{2} k_2\alpha \lambda (q_v(t_k) - \delta_v(t_k)) \right)
\end{align*}
\]
Note that since $B^i$ is bounded (see (9)), Theorem II.1 in [14] applies to the nonlinear time-varying hybrid system (14) (19); consequently, $(q_v, \omega, \delta_v, \hat{\delta}_4) = (0, 0, 0, 0)$ is a locally exponentially stable equilibrium for (14) (19) if and only if it is an exponentially stable equilibrium for (20) (21).

Now consider the continuous-time system (20) and sample its state $[q_v(t)^T \omega(t)^T]^T$ at $t = t_k$. Setting $q^*_v(k) \triangleq q_v(t_k)$, $\omega^*(k) \triangleq \omega(t_k)$, $m^*_{\text{coils}}(k) \triangleq m_{\text{coils}}(t_k)$, the following discrete-time system is obtained

$$
q^*_v(k + 1) = q^*_v(k) + \frac{1}{2} a(k, \Delta_m, \Delta_a) \omega^*(k) - J^{-1} G_1(k, \Delta_m, \Delta_a) m^*_{\text{coils}}(k) \quad k = 0, 1, 2, \ldots
$$

$$
\omega^*(k + 1) = \omega^*(k) - J^{-1} G_2(k, \Delta_m, \Delta_a) m^*_{\text{coils}}(k)
$$

where

$$
a(k, \Delta_m, \Delta_a) \triangleq t_{k+1} - t_k
$$

$$
G_1(k, \Delta_m, \Delta_a) \triangleq \int_{t_k}^{t_{k+1}} \frac{1}{2} (t_{k+1} - \tau) B^i(\tau)^\times d\tau
$$

$$
G_2(k, \Delta_m, \Delta_a) \triangleq \int_{t_k}^{t_{k+1}} B^i(\tau)^\times d\tau
$$

Note that the explicit dependence of $a$, $G_1$, and $G_2$ from $k$, $\Delta_m$, and $\Delta_a$ can be obtained by replacing $t_{k+1}$ and $t_k$ with the corresponding expressions that can be derived using (18). Defining $B^{\ast\ast}(k) \triangleq B^i(t_k)$, $\delta^*_v(k) \triangleq \delta_v(t_k)$, $\hat{\delta}_4^*(k) \triangleq \hat{\delta}_4(t_k)$ and adopting the previously introduced notation, output-feedback (21) reads as

$$
\delta^*_v(k + 1) = \delta^*_v(k) + p(k) \Delta_o (q^*_v(k) - \epsilon \lambda \delta^*_v(k))
$$

$$
\hat{\delta}_4^*(k + 1) = \hat{\delta}_4^*(k) - p(k) \Delta_o \epsilon \lambda \delta^*_v(k)
$$

$$
m^*_{\text{coils}}(k) = p(k) (B^{\ast\ast}(k)^\times)^T \epsilon^2 \left[ k_1 q^*_v(k) + \frac{1}{2} k_2 \alpha \lambda (q^*_v(k) - \epsilon \lambda \delta^*_v(k)) \right]
$$

By a simple adaptation of Proposition 7 in [15], it follows that if the linear time-varying discrete-time system (22) (26) is exponentially stable, then the linear time-varying hybrid system (20) (21) is exponentially stable, too.

Based on the previous discussion, our objective has become studying stability of linear time-varying discrete-time system (22) (26). For that purpose it is useful to perform the following change of variables $z_1(k) = q^*_v(k)$, $z_2(k) = \omega^*(k)/\epsilon$, $z_3(k) = q^*_v(k) - \epsilon \lambda \delta^*_v(k)$, $z_4(k) = \hat{\delta}_4^*(k)$
obtaining

\[
\begin{align*}
    z_1(k+1) &= z_1(k) + \frac{1}{2}a(k, \Delta_m, \Delta_a)z_2(k) - p(k)\epsilon^2 J^{-1}G_1(k, \Delta_m, \Delta_a)(B^i(k)^\times)^T \\
    &\quad \left(k_1z_1(k) + \frac{1}{2}k_2\alpha\lambda z_3(k)\right) \\
    z_2(k+1) &= z_2(k) - \epsilon p(k)J^{-1}G_2(k, \Delta_m, \Delta_a)(B^i(k)^\times)^T \left(k_1z_1(k) + \frac{1}{2}k_2\alpha\lambda z_3(k)\right) \\
    z_3(k+1) &= z_3(k) + \epsilon \left(\frac{1}{2}a(k, \Delta_m, \Delta_a)z_2(k) - \alpha\lambda p(k)\Delta z_3(k)\right) \\
    &\quad - \epsilon^2 p(k)J^{-1}G_1(k, \Delta_m, \Delta_a)(B^i(k)^\times)^T \left(k_1z_1(k) + \frac{1}{2}k_2\alpha\lambda z_3(k)\right) \\
    z_4(k+1) &= z_4(k) - \epsilon \Delta \alpha \lambda p(k)z_4(k)
\end{align*}
\]

(27)

Let

\[
    M(k, \Delta_m, \Delta_a) \triangleq p(k)G_2(k, \Delta_m, \Delta_a)(B^i(k)^\times)^T
\]

(28)

and consider the following averages

\[
\begin{align*}
    a_{av}(s, \Delta_m, \Delta_a) &\triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{k=s+1}^{k=s+N} a(k, \Delta_m, \Delta_a) \\
    M_{av}(s, \Delta_m, ) &\triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{k=s+1}^{k=s+N} M(k, \Delta_m, \Delta_a)
\end{align*}
\]

with \( s = 0, 1, 2, \ldots \). Both averages have been computed symbolically using Mathematica\textsuperscript{TM}. It turns out that \( a_{av} \) is independent of \( s \) and is equal to

\[
a_{av}(\Delta_m, \Delta_a) = \frac{\Delta_m + \Delta_a}{2}
\]

In addition, it is obtained that \( M_{av} \) is independent of both \( s \) and \( \Delta_m \). Its long expression is here omitted to save space. Moreover, from equation (25), it is easy to see that for \( k \) odd \( G_2(k, \Delta_m, 0) = 0 \); thus, taking into account that for \( k \) even \( p(k) = 0 \), from (28) we obtain that \( M(k, \Delta_m, 0) = 0 \). Then, it occurs that \( M_{av}(\Delta_a) \) vanishes at \( \Delta_a = 0 \), and consequently it can be expressed as

\[
    M_{av}(\Delta_a) = L_{av}(\Delta_a)\Delta_a
\]

(29)

where

\[
    L_{av}(\Delta_a) \triangleq \int_0^1 \frac{\partial M_{av}}{\partial \Delta_a}(u\Delta_a)du
\]

The expression of \( L_{av} \) has also been computed symbolically using Mathematica\textsuperscript{TM}, but again here it is omitted to save space since it is quite long. It turns out that \( L_{av}^0 \triangleq \lim_{\Delta_a \to 0} L_{av}(\Delta_a) \) is symmetric, and that matrix \( L_{av}^0 \) is positive definite if and only if the orbit is not equatorial, that is if its inclination satisfies \( incl \neq 0 \). Thus, we make the following hypothesis.
Assumption 1. The spacecraft’s orbit satisfies condition $L_{av}^0 > 0$.

Remark 1. In the special case of polar orbit ($incl = 90^\circ$) with zero RAAN ($\Omega = 0$), the expression of $L_{av}$ simplifies as follows

$$L_{av}(\Delta_a) = \frac{\mu^2}{32R^6} \begin{bmatrix} 4 + \frac{9\sin(2n\Delta_a)}{n\Delta_a} & 0 & -\frac{18\sin(n\Delta_a)^2}{n\Delta_a} \\ 0 & 2 \left(2 + \frac{9\sin(2n\Delta_a)}{n\Delta_a}\right) & 0 \\ \frac{18\sin(n\Delta_a)^2}{n\Delta_a} & 0 & \frac{9\sin(2n\Delta_a)}{n\Delta_a} \end{bmatrix}$$

The general fact that with $incl \neq 0$, matrix $L_{av}^0 \triangleq \lim_{\Delta_a \to 0} L_{av}(\Delta_a)$ is symmetric and positive definite can be easily checked in this special case.

At this point, consider the average system of (27) as defined in [16] which is given by

$$
\begin{align*}
z_1(k + 1) &= z_1(k) + \epsilon \frac{\Delta_m + \Delta_a}{4} z_2(k) \\
z_2(k + 1) &= z_2(k) - \epsilon \Delta_a J^{-1} L_{av}(\Delta_a) \left(k_1 z_1(k) + \frac{1}{2} k_2 \alpha \lambda z_3(k)\right) \\
z_3(k + 1) &= z_3(k) + \epsilon \frac{\Delta_m + \Delta_a}{2} \left(\frac{1}{2} z_2(k) - \alpha \lambda z_3(k)\right) \\
z_4(k + 1) &= z_4(k) - \epsilon \frac{\Delta_m + \Delta_a}{2} \alpha \lambda z_4(k)
\end{align*}
$$

It can be verified that all assumptions of [16, Theorem 2.2.2] are fulfilled by system (27). Thus, the following proposition holds true.

Proposition 1. If the average system (31) is exponentially stable for all $0 < \epsilon \leq \epsilon_0$, then there exists $0 < \epsilon_1 \leq \epsilon_0$, such that system (27) is exponentially stable for all $0 < \epsilon \leq \epsilon_1$.

Rewrite (31) in the following compact form

$$z(k + 1) = z(k) + \epsilon \Delta_a A(\Delta_m, \Delta_a) z(k)$$

with

$$A(\Delta_m, \Delta_a) = A_1(\Delta_a) + \frac{\Delta_m}{\Delta_a} A_2$$

where

$$A_1(\Delta_a) = \begin{bmatrix}
0 & \frac{1}{4} I & 0 & 0 \\
-k_1 J^{-1} L_{av}(\Delta_a) & 0 & -\frac{1}{2} k_2 \alpha \lambda J^{-1} L_{av}(\Delta_a) & 0 \\
0 & \frac{1}{4} I & -\frac{1}{2} \alpha \lambda I & 0 \\
0 & 0 & 0 & -\frac{1}{2} \alpha \lambda
\end{bmatrix}$$
and

\[
A_2 = \begin{bmatrix}
0 & \frac{1}{4}I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \frac{1}{4}I & -\frac{1}{2}\alpha\lambda I & 0 \\
0 & 0 & 0 & -\frac{1}{2}\alpha\lambda
\end{bmatrix}
\]  \hspace{1cm} (35)

For matrix \( A_1 \) the following lemma holds true.

**Lemma 1.** Assume that \( k_1 > 0, k_2 > 0, \alpha > 0, \lambda > 0 \) and Assumption 1 is satisfied. Then there exists \( \Delta_a^* > 0 \) such that \( A_1(\Delta_a) \) is a Hurwitz matrix for all \( 0 < \Delta_a < \Delta_a^* \).

**Proof.** First, it will be shown that \( A_0^1 \equiv \lim_{\Delta a \to 0} A_1(\Delta a) \) is a Hurwitz matrix. For that purpose it is useful to introduce the continuous-time system \( \dot{w} = A_0^1 w \) which in expanded form reads as follows

\[
\begin{align*}
\dot{w}_1 &= \frac{1}{4} w_2 \\
J \dot{w}_2 &= -L_{av}^0 \left( k_1 w_1 + \frac{1}{2} k_2 \alpha \lambda w_3 \right) \\
\dot{w}_3 &= \frac{1}{4} w_2 - \frac{1}{2} \alpha \lambda w_3 \\
\dot{w}_4 &= -\frac{1}{2} \alpha \lambda w_4
\end{align*}
\]  \hspace{1cm} (36)

Introduce the following Lyapunov function for (36)

\[
V_3(w) = 2 k_1 w_1^T L_{av}^0 w_1 + \frac{1}{2} w_2^T J w_2 + k_2 \alpha \lambda w_3^T L_{av}^0 w_3 + \frac{1}{2} w_4^2
\]

then

\[
\dot{V}_3(w) = -k_2 \alpha^2 \lambda^2 w_3^T L_{av}^0 w_3 - \frac{1}{2} \alpha \lambda w_4^2
\]

By La Salle’s invariance principle [17, Theorem 4.4], it follows that (36) is exponentially stable, and thus \( A_0^1 \) is Hurwitz. Then, by continuity of \( A_1(\Delta_a) \) with respect to \( \Delta_a \) the thesis follows immediately. \( \square \)

**Remark 2.** The value of \( \Delta_a^* \) can be determined numerically; in fact, once we fix the orbital parameters \((R, \text{incl}, \Omega)\) which appear in \( L_{av}(\Delta_a) \), the inertia matrix \( J \), and the controller’s gains \( k_1, k_2, \alpha, \) and \( \lambda \), it suffices studying the behavior with \( \Delta_a \) of the maximum of the real parts of the eigenvalues of \( A_1(\Delta_a) \).

For average system (32) the following stability result holds true.

**Theorem 1.** Assume that Assumption 1 is satisfied. First fix \( k_1 > 0, k_2 > 0, \alpha > 0, \lambda > 0, \) then fix \( 0 < \Delta_a < \Delta_a^* \). Consider symmetric matrix \( P_1 > 0 \) such that

\[
P_1 A_1(\Delta_a) + A_1(\Delta_a)^T P_1 = -I
\]  \hspace{1cm} (37)
and let

$$\Delta_m^* \triangleq \frac{\Delta_a}{\|P_1A_2 + A_2^TP_1\|} \quad (38)$$

If $0 < \Delta_m < \Delta_m^*$, then, there exists $\epsilon_0 > 0$, such that system (32) is exponentially stable for all $0 < \epsilon \leq \epsilon_0$.

**Proof.** First it will be shown that picking $k_1 > 0$, $k_2 > 0$, $\alpha > 0$, $\lambda > 0$, $\Delta_a$ and $\Delta_m$ as stated in the hypothesis, it follows that matrix $A(\Delta_m, \Delta_a)$ (see (33)) is Hurwitz. Note that the choice of $k_1 > 0$, $k_2 > 0$, $\alpha > 0$, $\lambda > 0$, and $\Delta_a$ guarantees that $A_1(\Delta_a)$ is Hurwitz; then, consider the continuous time system

$$\dot{w} = A(\Delta_m, \Delta_a)w = \left( A_1(\Delta_a) + \frac{\Delta_m}{\Delta_a} A_2 \right)w$$

with the candidate Lyapunov function $V_2(w) = w^TP_1w$. Since $P_1$ is the solution of (37) we obtain

$$V_2(w) = w^T \left( P_1A_1(\Delta_a) + A_1(\Delta_a)^TP_1 + \frac{\Delta_m}{\Delta_a} (P_1A_2 + A_2^TP_1) \right)w \leq - \left( 1 - \frac{\Delta_m}{\Delta_m^*} \right) \|w\|^2$$

Thus, if $0 < \Delta_m < \Delta_m^*$, it follows that $V_2$ is negative definite and consequently $A(\Delta_m, \Delta_a)$ is Hurwitz. Then, consider symmetric matrix $P > 0$ such that

$$PA(\Delta_m, \Delta_a) + A(\Delta_m, \Delta_a)^TP = -I \quad (39)$$

and let $V_3(z) = z^TPz$ be a candidate Lyapunov function for discrete-time system (32). Then, the following holds

$$\Delta V_2(z) \triangleq (z + \epsilon \Delta_a A(\Delta_m, \Delta_a)z)^TP(z + \epsilon \Delta_a A(\Delta_m, \Delta_a)z) - z^TPz$$

$$\leq \epsilon \Delta_a (-1 + \epsilon \Delta_a \|A(\Delta_m, \Delta_a)^TPA(\Delta_m, \Delta_a)\|) \|z\|^2$$

As a result, setting

$$\epsilon_0 = \frac{1}{2\Delta_a \|A(\Delta_m, \Delta_a)^TPA(\Delta_m, \Delta_a)\|} \quad (40)$$

we obtain that system (32) is exponentially stable for all $0 < \epsilon \leq \epsilon_0$.

Based on the previous theorem, first fix $k_1 > 0$, $k_2 > 0$, $\alpha > 0$, $\lambda > 0$, then determine $\Delta_a$ and $\Delta_m$ as indicated in the theorem, finally consider the corresponding value of $\epsilon_0$ given by (40); from Proposition 1 it follows that there exists $0 < \epsilon_1 \leq \epsilon_0$ such that system (27) is exponentially stable for all $0 < \epsilon \leq \epsilon_1$. In conclusion, from Theorem 1, Proposition 1, and the preceding discussion, the following main proposition is obtained.
Proposition 2. Consider the magnetically actuated spacecraft described by (8) and apply output-feedback (12) with $k_1 > 0$, $k_2 > 0$, $\alpha > 0$, $\lambda > 0$. Then, under Assumption 1, there exists $\Delta_a^* > 0$ such that fixed $0 < \Delta_a < \Delta_a^*$ and fixed $0 < \Delta_m < \Delta_m^*$ with $\Delta_m^*$ given by (38), there exists $\epsilon_1 \leq \epsilon_0$, with $\epsilon_0$ given by (40), such that for all $0 < \epsilon \leq \epsilon_1$ equilibrium $(q, \omega, \delta) = (\bar{q}, 0, \frac{1}{\alpha\lambda}\bar{q})$ is locally exponentially stable for (8) (12).

Remark 3. Usually in practical situations, time $\Delta_m$ is not a design parameter for the control engineer; in fact, as explained in Section 2., $\Delta_m$ is basically determined by the time that coils take to dissipate existing currents once the latter are switched off; as a result, $\Delta_m$ is a parameter intrinsic to the controlled plant and not a controller’s parameter. As a consequence, it might seem that the previous proposition is not useful for control design since it seems to imply that $\Delta_m$ can be chosen by the control designer. Actually, the previous proposition can be used to do control design as follows. First fix gains $k_1 > 0$, $k_2 > 0$, $\alpha > 0$, and $\lambda > 0$; then, determine $\Delta_a^*$ by using Lemma 1; for each $0 < \Delta_a < \Delta_a^*$ determine the corresponding $\Delta_m^*$ (see Theorem 1). By plotting $\Delta_m^*$ versus $\Delta_a$, we can graphically determine the set $S$ of values of $\Delta_a$ for which it occurs that $\Delta_m^* > \Delta_m$. Thus, fix $\Delta_a \in S$, and finally choose $\epsilon \leq \epsilon_0$ sufficiently small so that stability is achieved. In the next section, the described designed procedure will be illustrated through a case study.

Remark 4. The obtained stability results hold even if saturation on magnetic dipole moments is taken into account by replacing control (12) with

\[
\delta((j + 1)\Delta + \Delta_m) = \delta(j\Delta + \Delta_m) + \Delta_\alpha(q(j\Delta + \Delta_m) - \epsilon\lambda\delta(j\Delta + \Delta_m))
\]

\[
m_{\text{coils}}(t) = \begin{cases} 
0 & j\Delta \leq t < j\Delta + \Delta_m \\
\frac{1}{m_{\text{coils max}}} \text{sat} \left( \frac{1}{m_{\text{coils max}}} \right) (B'(q(j\Delta + \Delta_m), j\Delta + \Delta_m))^T \\
\epsilon^2 (k_1 q_v(j\Delta + \Delta_m) + k_2 \alpha \lambda W(q(j\Delta + \Delta_m)))^T \\
\left( (q(j\Delta + \Delta_m) - \epsilon\lambda\delta(j\Delta + \Delta_m))) \right) & j\Delta + \Delta_m \leq t < (j + 1)\Delta 
\end{cases}
\]

\[j = 0, 1, 2, \ldots \]  

where $m_{\text{coils max}}$ is the saturation limit for each dipole moment, and sat : $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the standard saturation function defined as follows; given $x \in \mathbb{R}^3$, the $i$-th component of sat($x$) is equal to $x_i$ if $|x_i| \leq 1$, otherwise it is equal to either 1 or -1 depending on the sign of $x_i$. The previous results still hold because saturation does not modify linearization (21).

4. Case Study

We consider a spacecraft whose inertia matrix is given by

\[
J = \begin{bmatrix} 5 & -0.1 & -0.5 \\
-0.1 & 2 & 1 \\
-0.5 & 1 & 3.5 \end{bmatrix} \text{ kg m}^2
\]
The inclination of the orbit is given by $incl = 87^\circ$, and the orbit’s altitude is 450 km; the corresponding orbital period is about 5600 s (see [18]). Without loss of generality the value $\Omega$ of RAAN is set equal to 0, whereas the initial phase $\phi_0$ (see (10)) has been randomly selected equal to $\phi_0 = 0.94$ rad.

Consider an initial state characterized by attitude equal to to the target attitude $q(0) = \bar{q}$, and by the following high initial angular rate

$$\omega(0) = [0.02 \ 0.02 \ -0.03]^T \text{rad/s} \quad (43)$$

due for example to an impact with an object. Moreover, consider that that the time needed for the spacecraft to perform all required measurements is equal to $\Delta_m = 10^{-3}$ s.

In [19] the continuous-time feedback (11) has been designed setting $k_1 = 7 \times 10^8$, $k_2 = 10^{10}$, $\epsilon = 10^{-3}$, $\alpha = 1 \times 10^3$, $\lambda = 1$. Here we design feedback (12) by keeping $k_1 = 7 \times 10^8$, $k_2 = 10^{10}$, $\alpha = 1 \times 10^3$, $\lambda = 1$ and by choosing parameters $\Delta_a$ and $\epsilon$ as follows. First, studying numerically the behavior with $\Delta_a$ of the maximum of the real parts of the eigenvalues of $A_1(\Delta_a)$ (see (34)), the plot in Fig. 2 is obtained; from the latter plot we determine the value $\Delta_a^* = 1197$ s for

![Figure 2. Maximum of the real parts of the eigenvalues of $A_1(\Delta_a)$.](image)

which it occurs that $A_1(\Delta_a)$ is Hurwitz for all $0 < \Delta_a < \Delta_a^*$. Then, for each $0 < \Delta_a < \Delta_a^*$ we determine the value $\Delta_m^*$ given by (38) obtaining the plot in Fig. 3. Recall that for each $\Delta_a$, magnitude $\Delta_m^*$ represents the maximum value for $\Delta_m$ for which stability can be achieved. Then, since in our case study $\Delta_m$ is fixed and equal to $10^{-3}$ s, the set $S$ of values of $\Delta_a$ for which stability is achievable is determined considering those values for which $\Delta_m^*$ is greater than $10^{-3}$.
s. From Fig. 3 it can be obtained that stability can be achieved for all \( 17.97 \, \text{s} < \Delta_a < 1191 \, \text{s} \). Then, we pick \( \Delta_a = 18 \, \text{s} \) to which there corresponds the value \( \epsilon_0 = 3.65 \times 10^{-5} \) (see (40)). By Proposition 2, it occurs that setting \( \epsilon \leq 3.65 \times 10^{-5} \) and small enough, equilibrium \((\bar{q}, \omega, \delta) = (\bar{q}, 0, \frac{1}{\epsilon} \bar{q})\) is locally exponentially stable for (8) (12). Then, fix \( \epsilon = 3.65 \times 10^{-5} \) obtaining the simulation results reported in Fig. 4. The plots show that the desired attitude is achieved in about 300 orbits. It turns out that the upper bound \( \epsilon_0 = 3.65 \times 10^{-5} \) for \( \epsilon \) is quite conservative. In fact, setting \( \epsilon = 10^{-4} \), the time behaviors reported in Fig. 5 are obtained, and they show that the desired attitude is now acquired in only 90 orbits.

In Fig. 6 the time behavior of \( m_{\text{coils}} \) restricted to the time interval \([0 \ 2] \) \( \times 10^{-2} \) s is plotted in order to display clearly the presence of the measurement interval \([0 \ \Delta_m = 10^{-3}] \) s in which \( m_{\text{coils}} = 0 \).

5. Conclusion

In this work a magnetic control law for spacecraft attitude stabilizations has been presented; the control law possess the important feature of separating measurement time from actuation time and does not require measures of attitude rate. A design method for the parameters of the controller has been obtained, and it has been successfully applied to a case study.

Acknowledgments

The author acknowledges Prof. A. Nascetti for fruitful discussions.
References


Figure 4. Simulations with $\epsilon = 3.65 \times 10^{-5}$. 
Figure 5. Simulations with $\epsilon = 10^{-4}$. 
Figure 6. Time behavior of $m_{\text{coils}}$ restricted to the interval $[0 \ 0.01]$ s.