Output Stabilization of Strongly Minimum-phase Systems

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Abstract: In this work it is shown that adapting the notions of relative degree and strong minimum-phasesness introduced in Liberzon et al. IEEE Trans. Automat. Control 47 (3) 2002, it is possible to obtain global stabilization results via output feedback. The notions in question are coordinate-free; consequently, it is not required that the system is transformed into normal form. The stabilization methods are based on small-gain arguments and provide global results.

Keywords: output-input stability, minimum-phase systems, relative degree, stabilization, output feedback.

1. INTRODUCTION

The notion of a minimum-phase system plays a role of paramount importance in feedback design, of linear as well as of nonlinear systems. For nonlinear systems that are affine in the control, the minimum-phase property has been defined in Byrnes and Isidori [1988] in terms of the asymptotic properties of its zero dynamics, that is the internal dynamics forced by the input that holds the output constantly at zero. Minimum-phase systems, according to Byrnes and Isidori [1988], are those systems in which the zero dynamics possess a globally asymptotically stable equilibrium. One of the major outcomes of this definition is that minimum-phase systems that are affine in the control and possess a globally defined normal form can be semi-globally practically stabilized by means of (a possibly dynamic, if the relative degree is higher than 1) output feedback. Other major applications of this concept are found in a series of papers dealing with the problem of asymptotic tracking/rejection of inputs/disturbances generated by an autonomous exosystem (see, one for all, the very recent paper Marconi et al. [2010]). An extension of the concept of minimum-phase system as defined in Byrnes and Isidori [1988] has been proposed and discussed in Liberzon et al. [2002], with various purposes. One purpose was the interest in offering an extension of the notion of relative degree for systems which are not affine in the control and do not possess a globally defined normal form. Another purpose was to provide a formal, and coordinate-free, characterization of the class of systems in which the state is bounded by a function of the outputs and its first \( r - 1 \) derivatives (with \( r \) being the relative degree), modulo a decaying term depending on the initial conditions. If a system possesses a globally defined normal form, the property in question is simply the property that the inverse dynamics is input-to-state stable (with the respect to the output and its \( r - 1 \) derivatives viewed as inputs), but the definition provided in Liberzon et al. [2002] is actually substantially more general, since it is not tied to the existence of a global normal form. Since the property in question implies, but in general is not implied by, the property of being minimum-phase in the sense of Byrnes and Isidori [1988], system possessing this property have been called strongly minimum-phase systems. Finally, the notion introduced in Liberzon et al. [2002] has interesting applications in the problem of asymptotic stabilization via state feedback and in extending to nonlinear systems a basic result from linear adaptive control.

As anticipated above, a relative degree one system possessing a global normal form and a globally asymptotically stable zero dynamics can be semiglobally practically (or even asymptotically, under appropriate additional hypotheses) stabilized by “high-gain” memoryless output feedback. This property is a simple consequence of the application of the (nonlinear) small gain theorem to a system expressed in normal form. The extension of this to systems having higher relative degree, which is not terribly difficult, requires dynamic output feedback and the use of a robust observer (as in Esfandiari and Khalil [1992] and Teel and Praly [1995]). Limitations of this design paradigm are the necessity of a globally defined normal form, on one hand, and the lack of guarantee of global convergence, on the other hand. Thus, one may wonder whether the stronger notion introduced in Liberzon et al. [2002], which is coordinate-free and is more “robust” with respect to the role played by the outputs when the latter are nonzero, would serve to the purpose of removing these limitations. In this paper, we show that, after some minor adaptation of the concepts proposed in Liberzon et al. [2002], this is actually the case.

In view of the possible application to problems of tracking/rejection, we consider in this paper the problem of output stabilization, which is defined as follows. Consider a nonlinear smooth system described by

\[
\dot{x} = f(x, u)
\]

with state \( x \in \mathbb{R}^n \) and control input \( u \in \mathbb{R} \). Associated with (1) there is a regulated output \( e \in \mathbb{R} \) expressed as
\[ e = h(x) \]  
(2) 
and a measured output \( y \in \mathbb{R}^p \) expressed as 
\[ y = h_m(x) \]  
(3) 
in which \( h : \mathbb{R}^n \to \mathbb{R} \) and \( h_m : \mathbb{R}^n \to \mathbb{R}^p \) are smooth functions. The control law for (1)–(3) is to be provided by a continuous system modeled by equations of the form 
\[ \dot{x}_c = f_c(x_c, y) \]  
\[ u = h_c(x_c, y) \]  
(4) 
with state \( x_c \in \mathbb{R}^{n_c} \).

In what follows we will say that the controller (4) solves the problem of global output stabilization, if the closed-loop system
\[ \dot{x} = f(x, h_c(x_c, h_m(x))) \]  
\[ \dot{x}_c = f_c(x_c, h_m(x)) \]  
(5)
has the following properties:
(i) the solutions of (5) are ultimately bounded i.e. there exists a bounded set \( B \subset \mathbb{R}^n \times \mathbb{R}^{n_c} \) with the property that, for every pair of compact sets \( X^\circ \subset \mathbb{R}^n \) and \( X^\circ_c \subset \mathbb{R}^{n_c} \), there is a time \( T > 0 \) such that \( (x(t), x_c(t)) \in B \) for all \( t \geq T \) and all \( (x(0), x_c(0)) \in X^\circ \times X^\circ_c \).
(ii) \( \lim_{t \to -\infty} h(x(t)) = 0 \ \forall (x(0), x_c(0)) \in \mathbb{R}^n \times \mathbb{R}^{n_c} \).

2. DEFINITIONS

2.1 Relative degree

The concept of “relative degree”, which is well understood and largely used in the analysis and design of input-affine SISO systems, can be extended in various ways to the case of systems in which \( f(x, u) \) is not affine in \( u \). An interesting extension of this concept has been presented in Liberzon et al. [2002], in the context of the general problem of establishing a connection between the properties of a system – of being “strong minimum-phase” and of being “output-input stable”. In what follows we propose a modification of the notion of “relative degree” of a system presented in Liberzon et al. [2002], which carries some interesting consequences in the context of the problem of achieving global stability via output feedback.

Set \( H_0(x) \triangleq h(x) \) and recursively define, as in Liberzon et al. [2002], a set of functions \( H_i : \mathbb{R}^n \times \mathbb{R}^i \to \mathbb{R} \), for \( i = 1, 2, \ldots \), as follows

\[ H_i(x, u_0, \ldots, u_{i-1}) \triangleq \frac{\partial H_{i-1}}{\partial x}(x, u_0, \ldots, u_{i-2})f(x, u_0) + \sum_{j=0}^{i-2} \frac{\partial H_{i-1}}{\partial u_j}(x, u_0, \ldots, u_{i-2})u_{j+1} . \]  
(6)

The importance of the functions \( H_i \) lies in the fact that if input \( u(t) \in C^{i-1} \) for some positive integer \( i \), then the corresponding regulated output \( y(t) \) has a continuous \( i \)-th derivative satisfying
\[ e^{(i)}(t) = H_i(x(t), u(t), \ldots, u^{(i-1)}(t)) . \]

In particular, if \( H_i \) is independent of \( u_0, \ldots, u_{i-1} \) for all \( i \) less than some positive integer \( r \), then \( H_{i} \) depends only on \( x \) and \( u_0 \). If this is the case, we simply write \( H_i(x) \) instead of \( H_i(x, u_0, \ldots, u_{i-1}) \) for \( i = 1, \ldots, r - 1 \) and
\[ H_r(x, u_0) = \frac{\partial H_{r-1}}{\partial x}(x)f(x, u_0) . \]

As an example, for the input-affine system
\[ \dot{x} = f(x) + g(x)u \],

if \( H_i \) is independent of \( u_0, \ldots, u_{i-1} \) for all \( i < r \) we have
\[ H_i(x) = L_i^j h(x) \quad \forall \ i < r \]  
\[ H_r(x, u_0) = L_r^j f(x) + L_{r-1}^j L_{r-1}^{-1} f(x) u_0 . \]

This being the case, we define the relative degree of a (possibly non-affine) system as follows.

**Definition 1.** A positive integer \( r \) is the (uniform) relative degree of system (1) if the following three conditions are satisfied

1. for each \( i < r \), the function \( H_i \) is independent of \( u_0, \ldots, u_{i-1} \);
2. there exist a class \( \mathcal{K}_\infty \) function \( \rho \) such that the following inequality holds for all \( x \in \mathbb{R}^n \) and all \( u_0 \in \mathbb{R} \)
\[ |u_0| \leq \rho(|H_r(x, u_0) - H_r(x, 0)|) ; \]
3. for all \( x \in \mathbb{R}^n \) and all \( u_0 \in \mathbb{R} \)
\[ (H_r(x, u_0) - H_r(x, 0))\operatorname{sgn}(u_0) \geq 0 . \]

As it is in the case of Liberzon et al., 2002, Definition 2], it is straightforward to see that if some \( r \) satisfies Definition 1, property (1) cannot hold for any \( r' > r \). This definition, however, is more suitable in the context of feedback stabilization. Consider, to this end, the two simple systems \( e = u^2 \) and \( e = u^3 \). Both systems have relative degree 1 in the sense of Liberzon et al., 2002, Definition 2, but only the latter has relative degree 1 in the sense of our Definition 1. Now, while in the latter the equilibrium \( e = 0 \) can be trivially made globally asymptotic stable by means of the feedback law \( u = -e \), the problem of asymptotically stabilizing the equilibrium \( e = 0 \) of the former system does not have a solution.

Moreover, in the special case in which \( H_r(x, u_0) \) is affine in \( u_0 \), that is it can be expressed as \( H_r(x, u_0) = a(x) + b(x)u_0 \), the definition of relative degree of Liberzon et al. [2002] implies \( b(x_0) \neq 0 \) for all \( x_0 \), which in turn implies \( H_r(0, 0) \neq 0 \) (see [Liberzon et al., 2002, Propostion 2]). However this is not the case with Definition 1 above. The property \( H_r(0, 0) = 0 \) is useful, but not necessary in the context of output stabilization; in fact, appealing to the results recently presented in Marconi et al. [2007] and Marconi et al. [2010], it is possible to show that the property in question can always be enforced in a properly “augmented” system. This confirms that the definition given above is more convenient in the context of the problem of feedback stabilization.

2.2 a-strongly minimum-phase systems

The definition which follows is very much related to the Definition 3 of Liberzon et al. [2002] of strongly minimum-phase system, which roughly speaking characterizes the class of those systems, having relative degree \( r \), in which the state of a system is bounded by a suitable function of the output and its first \( r-1 \) derivatives, modulo a decaying term depending on the initial conditions. For convenience, though, we express this definition in terms of asymptotic estimates, providing in this way a setup that is particularly suited to the analysis of certain interconnected system. This enables us to discuss stabilization by output feedback.

The property in question can always be enforced in a properly “augmented” system. This confirms that the definition given above is more convenient in the context of the problem of feedback stabilization.

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in terms which are essentially the same as those introduced in Teel [1996] to study stability of interconnected systems via the small-gain theorem. Moreover, for the sake of more generality, we consider the case in which systems are “strongly minimum-phase” with respect to compact sets. In this case, other than seeking estimates of the (euclidean) norm \( \|x(t)\| \) of the state \( x(t) \), we look for estimates of the distance \( d_A(x(t)) \) of \( x(t) \) from a compact set \( A \), in which \( d_A(x) = \min_{y \in A} \| x - y \| \).

Borrowing a few notations from Liberzon et al. [2002], we let \( \| \cdot \|_{[a,b]} \) denote the supremum norm of a signal restricted to an interval \([a,b]\). Moreover, given a \( \mathbb{R} \)-valued signal \( e \) and a nonnegative integer \( k \), we denote by \( e^k \) the \( \mathbb{R}^k \)-valued signal
\[
  e^k \triangleq \text{col}(e, e^{(1)}, \ldots, e^{(k-1)}) ,
\]
provided that the indicated derivatives exist.

Consider now a system having relative degree \( r \) and observe that, for \( i = 0, 1, \ldots, r - 1 \),
\[
e^i = h_i(x)
\]
in which
\[
h_i(x) \triangleq \text{col}(h_0(x), h_1(x), \ldots, h_i(x)) .
\]
Moreover, let \( Z \) denote the set
\[
Z \triangleq \{ x \in \mathbb{R}^n | h_0(x) = h_1(x) = \cdots = h_{r-1}(x) = 0 \} .
\]
The definition that follows expresses, as observed above, the property that the distance of \( x(t) \) from \( A \) is bounded by a suitable function of the output and its first \( r - 1 \) derivatives, modulo a decaying term depending on the initial conditions.

**Definition 2.** Consider a relative degree \( r \) system and let \( A \) be a compact subset of \( Z \). This system is \( a \)-strongly minimum-phase \(^1\) with respect to \( A \), if there exist \( \gamma^o \in \mathcal{K} \) and \( \gamma \in \mathcal{K}_\infty \) such that the following two conditions are satisfied

\[
(1) \quad \text{for every initial state } x(0) \in \mathbb{R}^n \text{ and every input } u(t) \in C^0 \text{ the corresponding solution satisfies the following inequality as long as it exists}
\]
\[
d_A(x(t)) \leq \max \{ \gamma^o (d_A(x(0))), \gamma (\|e^{r-1}(0)\|_{[0,t]} \} ; \quad (9)
\]
\[
(2) \quad \text{for every initial state } x(0) \in \mathbb{R}^n \text{ and every input } u(t) \in C^0 \text{ for which the corresponding } e^{r-1}(t) \text{ is bounded, the following inequality is satisfied}
\]
\[
\limsup_{t \to \infty} d_A(x(t)) \leq \gamma (\limsup_{t \to \infty} \|e^{r-1}(t)\|) . \quad (10)
\]

It is readily seen that if a system is strongly minimum-phase as in [Liberzon et al., 2002, Definition 3], then it is \( a \)-strongly minimum-phase with respect to \( \{0\} \), as shown in the following lemma.

**Lemma 3.** Consider a relative degree \( r \) system (1)-(2) with \( h(0) = 0 \). If the system is strongly minimum-phase (see Definition 3 of Liberzon et al. [2002]), then it is a strongly minimum phase with respect to \( \{0\} \).

**Proof.** First observe that \( h(0) = 0 \) implies that \( \{0\} \subseteq Z \). Then note that by Definition 3 of Liberzon et al. [2002], there exist functions \( \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K}_\infty \) such that for every initial state \( x(0) \in \mathbb{R}^n \) and every input \( u(t) \in C^0 \) the corresponding solution satisfies the following inequality as long as it exists
\[
|e(t)| \leq \beta(|e(0)|, t) + \gamma (\|e^{r-1}(0)\|_{[0,t]} \) . \quad (11)
\]

The previous equation yields
\[
|e(t)| \leq \beta(|e(0)|, t) + \gamma (\|e^{r-1}(0)\|_{[0,t]} \)
\]
which implies (9) with \( A = \{0\} \). Now assume that \( e^{r-1}(t) \) is bounded; then, (11) yields
\[
\limsup_{t \to \infty} |e(t)| \leq \gamma (\|e^{r-1}\|_{\infty}) . \quad (12)
\]

Let
\[
a = \limsup_{t \to \infty} \|e^{r-1}(t)\|
\]
and note that \( a \) is finite since \( e^{r-1}(t) \) is bounded. Pick \( \epsilon > 0 \) and let \( h > 0 \) be such that
\[
\gamma (a + h) - \gamma (a) < \epsilon .
\]

By definition of \( a \), there exists \( \bar{T} > 0 \) such that \( |e^{r-1}(t)| \leq a + h \forall t \geq \bar{T} \). Then \( \tilde{x}(t) \) denote the response of (1) from the initial state \( \tilde{x}(0) = x(\bar{T}) \) and input \( \tilde{u}(t) \) defined as
\[
\tilde{u}(t) = u(t + \bar{T}) .
\]

Clearly \( \tilde{x}(t) = x(t + \bar{T}) \), where \( x(t) \) was the response from the initial state \( x(0) \) and input \( u(t) \). Thus, \( e^{r-1}(t) = h_{r-1}(\tilde{x}(t)) = e^{r-1}(t + \bar{T}) \), and consequently, by definition of \( T \)
\[
|\tilde{e}^{r-1}(t)| \leq a + h \forall t \geq 0
\]
which implies \( \|\tilde{e}^{r-1}\|_{\infty} \leq a + h \). Then (12) implies
\[
\limsup_{t \to \infty} |\tilde{e}(t)| = \limsup_{t \to \infty} |\tilde{x}(t)| \leq \gamma (\|\tilde{e}^{r-1}\|_{\infty}) \leq \gamma (a + h) < \gamma (a) + \epsilon .
\]

Letting \( \epsilon \to 0 \) yields (10) with \( A = \{0\} \). \(<\)

3. GLOBAL STABILIZATION OF RELATIVE DEGREE ONE SYSTEMS

If system (1)-(2) has relative degree one, then
\[
\hat{e} = H_1(x, u) . \quad (13)
\]

For system (13) the following lemma holds.

**Lemma 4.** Consider a relative degree 1 system and let \( A \) be a compact subset of \( Z \). Suppose that \( H_1(x, 0) = 0 \) for all \( x \in A \). Then, for every \( \gamma \in \mathcal{K}_\infty \) there exists a continuous feedback law \( u = k^e(e) \) and a class \( \mathcal{K} \) function \( \gamma^o \) such that the resulting closed-loop system
\[
\hat{e} = H_1(x, k^e(e)) , \quad (14)
\]
viewed as a system with input \( x \) and state \( e \), has following properties:

\[
(1) \quad \text{for every initial state } e(0) \in \mathbb{R} \text{ and every input } x(t) \in C^0 \text{ the corresponding solution } e(t) \text{ satisfies the following inequality as long as it exists}
\]
\[
|e(t)| \leq \max \{ \gamma^o (|e(0)|), \gamma (\|d_A(x(0))\|_{[0,t]} \} ; \quad (15)
\]
\[
(2) \quad \text{for every initial state } e(0) \text{ and every bounded input } x(t) \in C^0 \text{, the corresponding solution } e(t) \text{ satisfies}
\]
\[
\limsup_{t \to \infty} |e(t)| \leq \gamma (\limsup_{t \to \infty} \|d_A(x(t))\|) . \quad (16)
\]

**Proof.** Choose
\[
k^e(e) = -\alpha(|e|) \text{sgn}(e)
\]

\(^1\) As in Teel [1996], the prefix “\( a \)” can be read for “asymptotically”.\]
with $\alpha \in K_\infty$. Let $V(e) = \frac{1}{2} e^2$; then,
\[
\frac{\partial V}{\partial e}(e) H_1(x, k'(e)) = H_1(x, 0) e + (H_1(x, k'(e)) - H_1(x, 0)) \text{sgn}(e)|e|.
\]
By (7) and (8) we obtain
\[
\alpha(|e|) \geq \rho \circ 2\sigma(d_A(x)) \quad \Rightarrow \quad \frac{\partial V}{\partial e}(e) H_1(x, k'(e)) \leq -\frac{1}{2} \rho^{-1}(\alpha(|e|))|e|.
\]
Choose
\[
\alpha(r) = \rho \circ 2\sigma \circ \gamma^{-1}(r).
\]
The previous choice of $\alpha$ makes the following property fulfilled
\[
|e| \geq \gamma(d_A(x)) \quad \Rightarrow \quad \frac{\partial V}{\partial e}(e) H_1(x, k'(e)) \leq -\frac{1}{2} \rho^{-1}(\alpha(|e|))|e|.
\]
In fact
\[
|e| \geq \gamma(d_A(x)) \quad \Rightarrow \quad \gamma^{-1}(|e|) \geq d_A(x) \quad \Rightarrow \quad \rho \circ 2\sigma \circ \gamma^{-1}(|e|) \geq \rho \circ 2\sigma(d_A(x)) \quad \Rightarrow \quad \alpha(|e|) \geq \rho \circ 2\sigma(d_A(x)).
\]
Then, the thesis is a consequence of (18) (see [Isidori, 1999, p. 19-21 and Theorem 10.4.5]). 

The main result of the section can now be stated.

**Theorem 5.** Consider a relative degree 1 system and let $A$ be a compact subset of $Z$. Suppose the system is a strongly minimum-phase with respect to $A$. Suppose that $H_1(x, 0) = 0$ for all $x \in A$. Then there exists a continuous feedback law $u = k'(e)$ such that, in the resulting closed-loop system $\dot{x} = f(x, k'(h(x)))$, any $x(0) \in \mathbb{R}^n$ produces trajectory that is bounded on $[0, \infty)$ and $\lim_{t \to \infty} d_A(x(t)) = 0$.

**Proof.** Appealing to Lemma 4, pick
\[
\gamma(r) = \frac{1}{2} \gamma^{-1}(r),
\]
where $\gamma$ characterizes the property of the system of being a strongly minimum phase, and let $k' \in C^0$ and $\gamma \in K$ be the resulting functions which make (15) - (16) fulfilled. Now consider the controlled plant
\[
\dot{x} = f(x, k'(h(x))).
\]
Pick an arbitrary $x(0) = x_0 \in \mathbb{R}^n$. First it will be shown that the corresponding trajectory $x(t)$ of (20) exists for all $t \geq 0$ and is bounded. By way of contradiction assume that for every $R > 0$ there exists $T > 0$ such that $x(t)$, defined on $[0, T]$, satisfies $d_A(x(T)) > R$. Let $e^o = h(x^o)$, and choose $R$ such that
\[
R > \max \left\{ \gamma^o(d_A(x^o)), \gamma \circ \gamma^o(|e^o|) \right\}.
\]
By Definition 2 (with $h = 1$)
\[
||d_A(x)||_{[0,T]} \leq \max \left\{ \gamma^o(d_A(x^o)), \gamma(|e||[0,T]) \right\}.
\]
Then, combining the above inequality with (15) yields
\[
||d_A(x)||_{[0,T]} \leq \max \left\{ \gamma^o(d_A(x^o)), \gamma \circ \gamma^o(|e^o|), \right\},
\]
\[
\gamma \circ \gamma^o(||d_A(x)||_{[0,T]}).
\]
Now note that the choice (19) implies
\[
\gamma \circ \gamma^o(|e|) < \rho \exists r > 0.
\]
Consequently, (23) simplifies to
\[
||d_A(x)||_{[0,T]} \leq \max \left\{ \gamma^o(d_A(x^o)), \gamma \circ \gamma^o(|e^o|) \right\}.
\]
Thus, using (21) obtain the contradiction
\[
d_A(x(T)) \leq \max \left\{ \gamma^o(d_A(x^o)), \gamma \circ \gamma^o(|e^o|) \right\} < R
\]
which allows us to conclude that $x(t)$ is defined for all $t \geq 0$ and is bounded.

In order to show that $\lim_{t \to \infty} d_A(x(t)) = 0$ proceed as follows. Since both $x(t)$ and $e(t)$ are bounded, then (10) holds (with $r = 1$), and (16) is satisfied. Thus, combining them, we obtain
\[
\limsup_{t \to \infty} d_A(x(t)) \leq \gamma \circ \gamma^o(\limsup_{t \to \infty} d_A(x(t))).
\]
Since $\limsup_{t \to \infty} d_A(x(t))$ is finite and (24) holds, we obtain $\lim_{t \to \infty} d_A(x(t)) = 0$.

**Remark 6.** As a direct consequence of the theorem, the memory-less feedback $u = k'(e)$ solves the global output stabilization problem for (1)-(2) with measure $y = e = h(x)$. In fact, property (i) is satisfied with any bounded set $B \supset A$ such that $\partial B \cap \partial A = \emptyset$, while property (ii) is implied by the fact that $A \subseteq Z \subseteq \{x \in \mathbb{R}^n | h(x) = 0\}$.

4. THE CASE OF HIGHER RELATIVE DEGREE

4.1 Partial-state feedback

It is well-known that, if a nonlinear input-affine system has relative degree $r > 1$, possesses a globally defined normal form, and has a globally asymptotically stable zero dynamics, semi-global asymptotic stability can be achieved by means of a “partial state” feedback, namely a memoryless feedback driven by the output and its first $r-1$ derivatives (see Teel and Praly [1995]). In this section we discuss the global, coordinate-free, version of this result.

Suppose system (1)-(2) has relative degree $r$. Let $K = (k_1, \ldots, k_{r-1}, 1) \in \mathbb{R}^r$ be a vector such that the polynomial
\[
s^{r-1} + k_{r-1}s^{r-2} + \ldots + k_2s + k_1
\]
is Hurwitz, and let the output $e$ of system (1) be replaced by the auxiliary “dummy” output
\[
\theta = h^\theta(x) = \sum_{i=1}^{r-1} k_i H_{r-1}(x) + H_{r-1}(x) = Kh_{r-1}(x).
\]
(26)

It is immediate to check that system (1)-(26) has relative degree one. In fact, setting $H^\theta_0(x) \triangleq h^\theta(x)$ and
\[
H^\theta_0(x, u_0) \triangleq \frac{\partial H^\theta_0}{\partial x}(x, u_0) f(x, u_0),
\]
it is easily seen that
\[ H_{\theta}^0(x, u_0) = \sum_{i=1}^{r-1} k_i H_i(x) + H_r(x, u_0). \] (27)
Thus
\[ H_{\theta}^0(x, u_0) - H_{\theta}^0(x, 0) = H_r(x, u_0) - H_r(x, 0), \]
and from this the thesis follows by straightforward use of Definition 1.

**Lemma 7.** Suppose system (1)-(2) has relative degree \( r \) and is a-strongly minimum-phase with respect to \( \mathcal{A} \), a compact subset of \( Z \). Then also system (1)-(26) is a-strongly minimum-phase with respect to \( \mathcal{A} \); in fact, there exist \( \gamma_{\theta}^0 \in \mathcal{K} \) and \( \gamma_{\theta} \in \mathcal{K}_\infty \) such that the following two conditions are satisfied

1. For every initial state \( x(0) \in \mathbb{R}^n \) and every input \( u(t) \in C^0 \) the corresponding solution satisfies the following inequality as long as it exists
   \[ d_A(x(t)) \leq \max\{\gamma_{\theta}(d_A(x(0))), \gamma_{\theta}(\|\theta\|_{[0,t]})) \} \] (28)
2. For every initial state \( x(0) \in \mathbb{R}^n \) and every input \( u(t) \in C^0 \) for which the corresponding \( \theta(t) \) is bounded, the following inequality is satisfied
   \[ \limsup_{t \to \infty} d_A(x(t)) \leq \gamma_{\theta}(\limsup_{t \to \infty} |\theta(t)|)). \] (29)

**Proof.** We already know that system (1)-(26) has relative degree one. Moreover, since \( x \in \mathcal{A} \subseteq Z \) implies \( H_i(x) = 0 \) for \( i = 0, \ldots, r-1 \), we see that \( \mathcal{A} \) is a compact subset of the set \( Z^\theta \triangleq \{ x \in \mathbb{R}^n | H^\theta_i(x) = 0 \} \). At this point, we must prove that system (1)-(26) satisfies the conditions in Definition 2 with \( r = 1 \) and \( \epsilon \) replaced by \( \theta \). To this end, note that equation (26) yields
\[ \theta(t) = e^{(r-1)t}(t) + \sum_{i=1}^{r-1} k_i e^{(1-i)t}(t). \] (30)

Now consider the following linear system with state \( \xi \in \mathbb{R}^{r-1} \) and input \( v \in \mathbb{R} \)
\[ \begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\cdots &
\xi_{r-2} &= -k_1 \xi_1 - \cdots - k_{r-1} \xi_{r-1} + v
\end{align*} \] (31)

Since (31) is asymptotically stable, there exist \( \gamma_{\xi}^2 > 0, \lambda > 0, \gamma_{\xi} > 0 \) such that for every initial state \( \xi(0) \in \mathbb{R}^{r-1} \) and every input \( v(t) \) the corresponding solution satisfies the following inequality as long as it exists
\[ |\xi(t)| \leq \max\{\gamma_{\xi}^2 e^{-\lambda t}|\xi(0)|, \gamma_{\xi} |v(t)|_{[0,t]} \}. \] (32)

Set \( K_0 = (k_1, \ldots, k_{r-1}) \) and observe that the vector
\[ z = \left( \begin{array}{c} \xi \\
-k_0 \xi + v \end{array} \right) \in \mathbb{R}^r \] (33)
satisfies
\[ |z| \leq |\xi| + | -K_0 \xi + v| \leq (1 + |K_0||\xi| + |v| \leq \max\{2(1 + |K_0||\xi|, 2|v| \}. \]

Thus, by using (32), we obtain that the following inequality holds as long as \( \xi(t) \) exists
\[ |\xi(t)| \leq \max\{\gamma_{\xi}^2 e^{-\lambda t}|\xi(0)|, \gamma_{\xi} |v(t)|_{[0,t]} \} \] (34)

where \( \gamma_{\xi}^2 = 2(1 + |K_0|)\gamma_{\xi}^2 \) and \( \gamma_{\xi} = 2(1 + |K_0|)\gamma_{\xi} + 2 \).

Set \( \xi_i(0) = e^{(i-1)t}(0) \) for \( i = 1, \ldots, r-1 \) and \( v(t) = \theta(t) \).

From (30) it is immediate to obtain that the corresponding solution of (31) is given by \( \xi_i(t) = e^{(i-1)t}(t) \) \( i = 1, \ldots, r-1 \), and thus from (30) and (33) \( \gamma_{\xi}(t) = -K_0 \xi(t) + \theta(t) = e^{(r-1)t}(t) \); consequently, from (34) we obtain that the following inequality is satisfied as long as it exists
\[ |e^{(r-1)t}| \leq \max\{2(1 + |K_0|)\gamma_{\xi}^2 e^{-\lambda t}|v(0)|, \gamma_{\xi} |v(t)|_{[0,t]} \}. \] (35)

which yields
\[ \|e^{(r-1)}\|_{[0,t]} \leq \max\{2(1 + |K_0|)\gamma_{\xi}^2 e^{-\lambda t}|v(0)|, \gamma_{\xi} |v(t)|_{[0,t]} \}. \] (36)

Combining (9) and (36) yields
\[ d_A(x(t)) \leq \max\{\gamma_{\xi}(d_A(x(0))), \gamma_{\xi}(\|\theta(0)\|_{[0,t]}), \gamma_{\xi}(\|\theta(t)\|_{[0,t]})) \}. \] (37)

Note that \( e^{-\lambda t} = h_{r-2}(x) \). Since \( h_{r-2} \) is smooth and vanishes on \( \mathcal{A} \), then there exist a function \( \gamma_{\theta} \in \mathcal{K}_\infty \) such that
\[ |h_{r-2}(x)| \leq \gamma_{\theta}(d_A(x)) \forall x \in \mathbb{R}^n. \] (38)

Thus, bearing in mind that \( e^{-\lambda t} = h_{r-2}(x(t)) \), equations (37) and (38) imply
\[ d_A(x(t)) \leq \max\{\gamma_{\theta}(d_A(x(0))), \gamma_{\theta}(\|\theta(0)\|_{[0,t]}), \gamma_{\theta}(\|\theta(t)\|_{[0,t]})) \}. \] (39)

with \( \gamma_{\theta}(r) = \max\{\gamma_{\theta}(r), \gamma_{\theta}(\gamma_{\xi}(h_{r-2}(r))) \} \) and \( \gamma_{\theta}(r) = \gamma_{\xi}(r) \).

Now assume that \( \theta(t) \) is bounded; then, by (39) \( x(t) \) is bounded too, and so is \( e^{-\lambda t} \); consequently, (10) holds. Moreover, it is known (see e.g. [Isidori, 1999, p. 28-30]) that inequality (35) yields
\[ \limsup_{t \to \infty} |e^{(r-1)t}| \leq \gamma_{\theta} \limsup_{t \to \infty} |\theta(t)|. \] (40)

Thus, from (10) and (40) it is immediate to obtain
\[ \limsup_{t \to \infty} d_A(x(t)) \leq \gamma_{\theta}(\limsup_{t \to \infty} |\theta(t)|). \] (41)

which completes the proof. \(<\)

The following output stabilization result can now be presented.

**Theorem 8.** Suppose system (1)-(2) has relative degree \( r \) and is a-strongly minimum-phase with respect to \( \mathcal{A} \), a compact subset of \( Z \). If \( H_i(x, 0) = 0 \forall x \in \mathcal{A} \), then there exists a continuous map \( k^0 : \mathbb{R} \to \mathbb{R} \) such that the feedback law \( u = k^0(K h_{r-1}(x)) \) solves the problem of global output stabilization.

**Proof.** By Lemma 7 and equation (27), it follows that system (1)-(26) satisfies all the assumptions of Theorem 5. Consequently, there exists a continuous feedback law \( u = k^0(\theta) \) such for the resulting closed-loop system
\[ \dot{x} = f(x, k^0(K h_{r-1}(x))) \] (42)

the following occurs: trajectories are ultimately bounded and \( \lim_{t \to \infty} d_A(x(t)) = 0 \). In particular \( \lim_{t \to \infty} e(t) = \lim_{t \to \infty} h(x(t)) = 0 \) which implies that conditions (i) and (ii) are satisfied. \(<\)

**Remark 9.** From the proofs of Lemma 4 and Theorem 5 it follows that
\[ k^0(\theta) = -\alpha_{\theta}(\|\theta\|)sgn(\theta), \]
where
\[ \alpha_{\theta}(r) = \rho \circ 2\sigma_\theta \circ \gamma_\theta(2r), \] (43)
and \( \sigma_\theta \) is any class \( \mathcal{K}_\infty \) function such that
\[ |H^0_1(x, 0)| \leq \sigma_\theta(d_A(x)) \forall x \in \mathbb{R}^n. \]
4.2 Output feedback

If the components of the “partial state” $h_{r-1}(x)$ are not available for measurement, one might seek to achieve global output stabilization by letting $h_{r-1}(x)$ be replaced by suitable asymptotic “estimates”, provided by the robust observer of Esfandiari and Khalil [1992] and Teel and Praly [1995]. To this end, though, some restrictive hypotheses are needed. One hypothesis is that the function $\gamma$ which characterizes the property of system (1)-(2) of being a strongly minimum phase is bounded by a linear function. The other hypotheses are the existence of a linearly bounded class $K_{\infty}$ function $\sigma_0$ such that

$$\max \left\{ \sum_{i=1}^{r} k_i H_i(x) + H_r(x,0) \mid |H_r(x,k^\theta(Kh_{r-1}(x)))| \right\} \leq \sigma_0(dA(x)) \quad \forall x \in \mathbb{R}^n$$

and of a linearly bounded class $K_{\infty}$ function $\ell$ such that

$$|H_r(x,k^\theta(K(h_{r-1}(x) + v))) - H_r(x,k^\theta(Kh_{r-1}(x)))| \leq \ell(|v|) \quad \forall x \in \mathbb{R}^n \quad \forall v \in \mathbb{R}^n.$$ 

If this is the case, it is possible to show that the required stabilization result can be obtained by means of a dynamic feedback law of the form

$$\begin{align*}
\dot{\xi}_1 &= \dot{\xi}_2 + g_{r-1}(h(x) - \xi_1) \\
\dot{\xi}_2 &= \frac{g'}{1} \dot{\xi}_r + g_{r-2}(h(x) - \xi_1) \\
&\vdots \\
\dot{\xi}_r &= \frac{g'}{1} \dot{\xi}_r + g_{r-1}(h(x) - \xi_1) \\
u &= k^\theta(K\xi).
\end{align*}$$

(44)

driven by the measured output $y = e = h(x)$.

A sketch of the proof consists in what follows. First of all, it is shown that under the said hypothesis the stabilizing law $u = k^\theta(Kh_{r-1}(x))$ is robust with respect to perturbations affecting $Kh_{r-1}(x)$. In other words it can be shown that system

$$\dot{x} = f(x,k^\theta(K(h_{r-1}(x) + v)))$$

is input-to-state stable with respect to the set $A_\epsilon$ with a linear gain function. The proof of this property uses arguments similar to those used in the proofs of Lemma 4 and Theorem 5. This being the case, let $c_0, c_1, \ldots, c_r$ be such that the polynomial

$$s^r + c_{r-1}s^{r-1} + \ldots + c_1s + c_0$$

is Hurwitz, and define

$$\epsilon = \xi - h_{r-1}(x).$$

Standard arguments prove that, under the said hypotheses, for every $L_{\epsilon} > 0$ there exists a number $g^*$ such that, for all $g \geq g^*$, the following estimates hold

$$|x(t)| \leq \beta_\epsilon(|e(0)|, t) + L_{\epsilon} dA(x) \quad \forall t \in [0,t]$$

for some $\beta_\epsilon \in K\mathcal{L}$. From this, the result follows by standard application of the small gain theorem.

5. CONCLUSIONS

We have shown that for a strongly minimum-phase systems it is possible to obtain global stabilization results via output feedback. Since the notion of a strongly minimum-phase is coordinate-free, then the proposed stabilization results do not require that the system is preliminarily transformed into normal form.